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Theory of q-Deformed Forms. I. q-Deformed Alternating Tensor and q-Deformed Wedge Product

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In this paper we deal with the q-deformed alternating tensor and prove the associativity of the q-deformed wedge product. Moreover, we construct the theory of q-deformed homology in order to prove the q-deformed Stokes theorem. Lastly we prove the q-deformed Poincaré lemma.

1. INTRODUCTION

Quantum groups provide a concrete example of noncommutative differential geometry (Connes, 1986). The idea of the quantum plane was first introduced by Manin (1988, 1989). The application of noncommutative differential geometry to quantum matrix groups was made by Woronowicz (1987, 1989). Wess and Zumino (1990; Zumino, 1991) considered one of the simplest examples of noncommutative differential calculus over Manin's quantum plane. They developed a differential calculus on the quantum hyperplane covariant with respect to the action of the quantum deformation of $GL(n)$, so-called $GL_q(n)$. Much subsequent work has been done in this direction (Schmidke *et al.*, 1989; Schirrmacher, 1991a,b; Schirrmacher *et al.*, 1991; Burdik and Hlavaty, 1991; Hlavaty, 1991; Burdik and Hellinger, 1992; Ubriaco, 1992; Giler *et al.*, 1991, 1992; Lukierski *et al.*, 1991; Lukierski and Nowicki, 1992; Castellani, 1992; Chaichian and Demichev, 1992; Chung, n.d.-a,b).

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It is of interest to examine whether this new mathematical object will bring new phenomena in the forthcoming physics. Since symmetries play an important role in physics, it is worth extending them to the deformed concept of symmetries which might be used in physics as well. If quantum groups are applied, they may be supposed to create a kind of new physics which goes back to its classical versions when the deformation parameters take particular values.

To this end it is worthwhile to consider the fundamental concepts and the computational techniques of quantum groups.

Recently the contraction rule of the q-deformed Levi-Civita symbol was treated (Chung *et al.*, n.d.). The present series of papers constitutes three parts.[2] In part I we deal with the q-deformed alternating tensor and prove the associativity of the q-deformed wedge product. Moreover, we construct the theory of q-deformed homology in order to prove the q-deformed Stokes theorem. Lastly we prove the q-deformed Poincaré lemma.

In part II we deal with the q-deformed differential forms and q-deformed Hamilton equation.

In part III we deal with the q-deformed Hodge dual operator and q-deformed self-dual Yang–Mills theory.

Although this paper (part I) is more or less mathematical, it is needed in constructing the q-deformed Hamilton equation (part II) and q-deformed self-dual Yang–Mills equation (part III). Since the notations given in this paper are compact, we first introduce special cases as exercises and then extend to the general case.

2. q-DEFORMED ALTERNATING TENSORS AND q-DEFORMED WEDGE PRODUCT

In this section we introduce the q-deformed alternating tensors and prove the associativity of the q-deformed wedge product. First we suggest the following definition. A tensor $\omega(v_1, v_2, \ldots, v_k)$ is called q-alternating if

$$\omega(\ldots, v_i, v_j, \ldots) = -q\omega(\ldots, v_j, v_i, \ldots) \qquad \text{for} \quad i > j \tag{1}$$

where vectors v_i and v_j are interchanged and all other v's are left fixed. The set of all q-alternating tensors is denoted $\Lambda_q^k(V)$. Here we define $Alt_q(T_q)$ by

$$Alt_q(T_q)(v_1, \ldots, v_k) = \frac{1}{[k]!} \sum_{\sigma \in S_k} sgn_q \sigma T_q(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \tag{2}$$

where

$$[k] = \frac{1 - q^{-k}}{1 - q^{-1}}, \qquad [k]! = [k][k-1] \ldots [2][1]$$

[2]The present paper is part I, Chung (1996a) is Part II, and Chung (1996b) is part III.

and S_k is the set of all permutations of the numbers 1 to k. The $sgn_q\sigma$ is defined as

$$sgn_q\sigma = q^{-R(\sigma(1),\ldots,\sigma(k))} E^{1,\ldots,k}_{\sigma(1)\ldots\sigma(k)}$$

where $R(\sigma(1), \ldots, \sigma(k))$ is defined as

$$R(\sigma(1), \ldots, \sigma(k)) = \sum_{n=1}^{k-1} \sum_{m=n+1}^{k} R(\sigma(n), \sigma(m))$$

and

$$R(i, j) = 1 \qquad \text{for} \quad i > j$$

$$R(i, j) = 0 \qquad \text{for} \quad i \leq j$$

and

$$E^{1,\ldots,k}_{\sigma(1)\ldots\sigma(k)} = \frac{E_{1\ldots k}}{E_{\sigma(1)\ldots\sigma(k)}}$$

Here $E_{i_1\ldots i_k}$ means the q-deformed Levi-Civita symbol (Chung *et al.*, n.d.), which is defined as follows:

$$E_{12\ldots k} = 1$$

$$E_{\ldots ij \ldots} = (-q)E_{\ldots ji \ldots} \qquad \text{for} \quad i > j$$

For example, the q-deformed q-alternating 2-tensor $T_q \in \Lambda^2_q(V)$ is defined as

$$\begin{aligned} &Alt_q(T_q)(v_1, v_2) \\ &= \frac{1}{[2]!} \{q^{-R(12)} E^{12}_{12} T_q(v_1, v_2) + q^{-R(21)} E^{12}_{21} T_q(v_2, v_1)\} \\ &= \frac{1}{[2]!} \{T_q(v_1, v_2) - q^{-2} T_q(v_2, v_1)\} \end{aligned}$$

Similarly we find

$$\begin{aligned} &Alt_q(T_q)(v_2, v_1) \\ &= \frac{1}{[2]!} \{q^{-R(12)} E^{21}_{12} T_q(v_1, v_2) + q^{-R(21)} E^{21}_{21} T_q(v_2, v_1)\} \\ &= \frac{1}{[2]!} \{-qT_q(v_1, v_2) + q^{-1} T_q(v_2, v_1) \end{aligned}$$

Here we can show that

$$Alt_q(T_q)(v_2, v_1) = -qAlt_q(T_q)(v_1, v_2)$$

which means that $Alt_q(T_q)$ is really q-alternating. The general proof is written in Theorem 1. For the case that $T_q \in \Lambda_q^3(V)$, we have the q-alternating tensor

$$\begin{aligned}
&Alt_q(T_q)(v_1, v_2, v_3)\\
&= \frac{1}{[3]!}\{q^{-R(123)}E_{123}^{123}T_q(v_1, v_2, v_3) + q^{-R(132)}E_{132}^{123}T_q(v_1, v_3, v_2)\\
&\quad + q^{-R(213)}E_{213}^{123}T_q(v_2, v_1, v_3)\\
&\quad + q^{-R(231)}E_{231}^{123}T_q(v_2, v_3, v_1) + q^{-R(312)}E_{312}^{123}T_q(v_3, v_1, v_2)\\
&\quad + q^{-R(321)}E_{321}^{123}T_q(v_3, v_2, v_1)\}\\
&= \frac{1}{[3]!}\{T_q(v_1, v_2, v_3) - q^{-2}T_q(v_1, v_3, v_2) - q^{-2}T_q(v_2, v_1, v_3)\\
&\quad + q^{-4}T_q(v_2, v_3, v_1) + q^{-4}T_q(v_3, v_1, v_2) - q^{-6}T_q(v_3, v_2, v_1)\}
\end{aligned}$$

Similarly we get

$$\begin{aligned}
&Alt_q(T_q)(v_1, v_3, v_2)\\
&= \frac{1}{[3]!}\{q^{-R(123)}E_{123}^{132}T_q(v_1, v_2, v_3) + q^{-R(132)}E_{132}^{132}T_q(v_1, v_3, v_2)\\
&\quad + q^{-R(213)}E_{213}^{132}T_q(v_2, v_1, v_3)\\
&\quad + q^{-R(231)}E_{231}^{132}T_q(v_2, v_3, v_1) + q^{-R(312)}E_{312}^{132}T_q(v_3, v_1, v_2)\\
&\quad + q^{-R(321)}E_{321}^{132}T_q(v_3, v_2, v_1)\}\\
&= \frac{1}{[3]!}\{-qT_q(v_1, v_2, v_3) + q^{-1}T_q(v_1, v_3, v_2) + q^{-1}T_q(v_2, v_1, v_3)\\
&\quad - q^{-3}T_q(v_2, v_3, v_1) - q^{-3}T_q(v_3, v_1, v_2) + q^{-5}T_q(v_3, v_2, v_1)\}
\end{aligned}$$

Here we can check that

$$Alt_q(T_q)(v_1, v_3, v_2) = -qAlt_q(T_q)(v_1, v_2, v_3)$$

The general extension is summarized in Theorem 1 below.

From the definition of $Alt_q(T_q)$, we reach the following theorem.

Theorem 1. $Alt_q(T_q) \in \Lambda_q^k(V)$.

Proof. Let $(i, i + 1)$ be the permutation that interchanges i and $i + 1$ and leave all other numbers fixed. If $\sigma \in S_k$, let $\sigma' = \sigma \circ (i, i + 1)$, which implies

$$\begin{aligned}
\sigma'(i) &= \sigma(i + 1)\\
\sigma'(i + 1) &= \sigma(i)\\
\sigma'(l) &= \sigma(l) \qquad \text{for} \quad l \neq i \text{ and } l \neq i + 1
\end{aligned}$$

Then

$$Alt_q(T_q)(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k)$$

$$= \frac{1}{[k]!} \sum_{\sigma \in S_k} sgn_q\sigma T_q(v_{\sigma(1)}, \ldots, v_{\sigma(i)}, v_{\sigma(i+1)}, \ldots, v_{\sigma(k)})$$

$$= \frac{1}{[k]!} \sum_{\sigma \in S_k} sgn_q\sigma T_q(v_{\sigma'(1)}, \ldots, v_{\sigma'(i+1)}, v_{\sigma'(i)}, \ldots, v_{\sigma'(k)})$$

$$= (-q)^{-1} \frac{1}{[k]!} \sum_{\sigma' \in S_k} sgn_q\sigma' T_q(v_{\sigma'(1)}, \ldots, v_{\sigma'(i+1)}, v_{\sigma'(i)}, \ldots, v_{\sigma'(k)})$$

$$= (-q)^{-1} Alt_q(T_q)(v_1, \ldots, v_{i+1}, v_i, \ldots, v_k) \tag{3}$$

where we used the relation

$$sgn_q\sigma = q^{-R(\sigma(1)\cdots\sigma(k))} E^{1,\ldots,i,i+1,\ldots,k}_{\sigma(1),\ldots,\sigma(i),\sigma(i+1),\ldots,\sigma(k)}$$

$$= q^{-R(\sigma'(1),\ldots,\sigma'(i+1),\sigma'(i),otso'(k))} E^{1,\ldots,i+1,\ldots,k}_{\sigma'(1),\ldots,\sigma'(i+1),\sigma'(i),\ldots,\sigma'(k)}$$

$$= (-q)^{-1} q^{-R(\sigma'(1),\ldots,\sigma'(i+1),\sigma'(i),otso'(k))} E^{1,\ldots,i+1,i,\ldots,k}_{\sigma'(1),\ldots,\sigma'(i+1),\sigma'(i),\ldots,\sigma'(k)}$$

$$= -q^{-1} sgn_q\sigma'$$

For $\omega \in \Lambda_q^2(V)$ we have

$$Alt_q\omega(v_1, v_2)$$

$$= \frac{1}{[2]!} \{\omega(v_1, v_2) - q^{-2}\omega(v_2, v_1)\}$$

$$= \frac{1}{[2]!} \{\omega(v_1, v_2) - q^{-2}(-q)\omega(v_1, v_2)\}$$

$$= \frac{1 + q^{-1}}{[2]!} \omega(v_1, v_2)$$

$$= (v_1, v_2)$$

Similarly we get

$$Alt_q\omega(v_2, v_1)$$

$$= \frac{1}{[2]!} \{-q\omega(v_1, v_2) + q^{-1}\omega(v_2, v_1)\}$$

$$= \frac{1}{[2]!} \{-q(-q^{-1})\omega(v_2, v_1) + q^{-1}\omega(v_2, v_1)\}$$

$$= \frac{1 + q^{-1}}{[2]!} \omega(v_2, v_1)$$

$$= (v_2, v_1)$$

Now let us check the case that $k = 3$. For $\omega \in \Lambda_q^3(V)$ we obtain

$$Alt_q\omega(v_1, v_2, v_3)$$
$$= \frac{1}{[3]!}\{\omega(v_1, v_2, v_3) - q^{-2}\omega(v_1, v_3, v_2) - q^{-2}\omega(v_2, v_1, v_3)$$
$$+ q^{-4}\omega(v_2, v_3, v_1) + q^{-4}\omega(v_3, v_1, v_2) - q^{-6}\omega(v_3, v_2, v_1)\}$$
$$= \frac{1 + 2q^{-1} + 2q^{-2} + q^{-3}}{[3]!}\omega(v_1, v_2, v_3)$$
$$= \omega(v_1, v_2, v_3)$$

Similarly we get

$$Alt_q\omega(v_1, v_3, v_2)$$
$$= \frac{1}{[3]!}\{-q\omega(v_1, v_2, v_3) + q^{-1}\omega(v_1, v_3, v_2) + q^{-1}\omega(v_2, v_1, v_3)$$
$$- q^{-3}\omega(v_2, v_3, v_1) - q^{-3}\omega(v_3, v_1, v_2) + q^{-5}\omega(v_3, v_2, v_1)\}$$
$$= \frac{1 + 2q^{-1} + 2q^{-2} + q^{-3}}{[3]!}\omega(v_1, v_3, v_2)$$
$$= \omega(v_1, v_3, v_2)$$

The general extension of the above computations are written in Theorem 2.

When ω is q-alternating, we have the following theorem.

Theorem 2. If $\omega \in \Lambda_q^k(V)$, then $Alt_q(\omega) = \omega$.

Proof. The proof is easy; we have

$$Alt_q(\omega)(v_1, \ldots, v_k)$$
$$= \frac{1}{[k]!}\sum_{\sigma \in S_k} sgn_q\sigma\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$
$$= \frac{1}{[k]!}\sum_{\sigma \in S_k} q^{-R(\sigma(1)\cdots\sigma(k))}E^{1\cdots k}_{\sigma(1)\cdots\sigma(k)}\omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)})$$
$$= \frac{1}{[k]!}\left(\sum_{\sigma \in S_k} q^{-R(\sigma(1)\cdots\sigma(k))}\right)\omega(v_1, \ldots, v_k)$$
$$= \omega(v_1, \ldots, v_k) \tag{4}$$

where we used the following identity:

$$\sum_{\sigma \in S_k} q^{-R(\sigma(1)\cdots\sigma(k))} = [k]!$$

This identity is easily proved by using the definition of $R(\sigma(1) \cdots \sigma(k))$, which is given in Appendix A.

From Theorems 1 and 2, we have the following theorem.

Theorem 3. $Alt_q(Alt_q(T_q)) = Alt_q(T_q)$.

Proof. The proof follows immediately from the proof of Theorems 1 and 2, so we will omit it here for brevity.

If $\omega \in \Lambda_q^k(V)$ and $\eta \in \Lambda_q^l(V)$, then $\omega \otimes \eta$ is usually not in $\Lambda_q^{k+l}(V)$. We will therefore define a new product, the q-wedge product $\omega \wedge_q \eta \in \Lambda_q^{k+l}(V)$, by

$$\omega \wedge_q \eta = \frac{[k + l]!}{[k]![l]!} Alt_q(\omega \otimes \eta) \tag{5}$$

where

$$\begin{aligned} &Alt_q(\omega \otimes \eta) \\ &= \frac{1}{[k+l]!} \sum_{\sigma \in S_{k+l}} sgn_q \sigma \omega(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \\ &\quad \times \eta(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \end{aligned} \tag{6}$$

The reason for the strange coefficient will appear later. The following properties of $\wedge_q$ hold:

$$(\omega_1 + \omega_2) \wedge_q \eta = \omega_1 \wedge_q \eta + \omega_2 \wedge_q \eta \tag{7}$$

$$\omega \wedge_q (\eta_1 + \eta_2) = \omega \wedge_q \eta_1 + \omega \wedge_q \eta_2 \tag{8}$$

$$a\omega \wedge_q \eta = \omega \wedge_q a\eta = a(\omega \wedge_q \eta), \qquad a \in R \tag{9}$$

$$(\omega \wedge_q \eta) \wedge_q \theta = \omega \wedge_q (\eta \wedge_q \theta) \tag{10}$$

The last property (10) will be proved later. From the definition of Alt_q, we have the following theorem.

Theorem 4. If $Alt_q(S_q) = 0$, then

$$Alt_q(S_q \otimes T_q) = Alt_q(T_q \otimes S_q) = 0$$

Proof. By definition we have

$$\begin{aligned} &[k + l]!\, Alt_q(S_q \otimes T_q)(v_1, \ldots, v_k, v_{k+1}, \ldots, v_{k+l}) \\ &= \sum_{\sigma \in S_{k+l}} sgn_q \sigma S_q(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) T_q(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \end{aligned}$$

If $G \subset S_{k+l}$ consists of all σ which leave $k+1, \ldots, k+l$ fixed, then

$$\sum_{\sigma \in G} sgn_q \sigma S_q(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) T_q(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

$$= \left[\sum_{\sigma \in S_k} sgn_q \sigma S_q(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \right] T_q(v_{k+1}, \ldots, v_{k+l})$$

$$= \left\{ [k]! \frac{1}{[k]!} \sum_{\sigma \in S_k} sgn_q \sigma S_q(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \right\} T_q(v_{k+1}, \ldots, v_{k+l})$$

$$= [k]!\, Alt_q(S_q)(v_1, \ldots, v_k) T_q(v_{k+1}, \ldots, v_{k+l})$$

$$= 0$$

where for $\sigma \in G$, we used

$$sgn_q \sigma \quad (\sigma \in G)$$

$$= q^{-R(\sigma(1),\ldots,\sigma(k),k+1,\ldots,k+l)} E^{1\cdots k k+1 \cdots k+l}_{\sigma(1)\cdots\sigma(k) k+1 \cdots k+l}$$

$$= q^{-R(\sigma(1)\cdots\sigma(k)) - R(k+1\cdots k+l) - \sum_{i \in \{\sigma(1),\ldots,\sigma(k)\}, j \in \{k+1,\ldots,k+l\}} R(i,j)} E^{1\cdots k}_{\sigma(1)\cdots\sigma(k)}$$

$$= q^{-R(\sigma(1)\cdots\sigma(k))} E^{1\cdots k}_{\sigma(1)\cdots\sigma(k)}$$

$$= sgn_q \sigma \quad (\sigma \in S_k) \tag{11}$$

Suppose now that σ_0 does not belong to G. Let $G \circ \sigma_0 = \{\sigma \circ \sigma_0 | \sigma \in G\}$ and let $v_{\sigma_0(1)}, \ldots, v_{\sigma_0(k+l)} = w_1, \ldots, w_{k+l}$.

Then we have

$$\sum_{\sigma \in G \circ \sigma_0} sgn_q \sigma S_q(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) T_q(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

$$= \sum_{\sigma' \in G} sgn_q(\sigma' \circ \sigma_0) S_q(v_{\sigma' \circ \sigma_0(1)}, \ldots, v_{\sigma' \circ \sigma_0(k)})$$

$$\times T_q(v_{\sigma' \circ \sigma_0(k+1)}, \ldots, v_{\sigma' \circ \sigma_0(k+l)})$$

$$= \sum_{\sigma' \in G} sgn_q(\sigma' \circ \sigma_0) S_q(\omega_{\sigma'(1)}, \ldots, \omega_{\sigma'(k)})$$

$$\times T_q(\omega_{k+1}, \ldots, \omega_{k+l})$$

$$= q^{R(\sigma_0(1),\ldots,\sigma_0(k+l))} sgn_q \sigma_0 \sum_{\sigma' \in S_k} sgn_q \sigma' S_q(\omega_{\sigma'(1)}, \ldots, \omega_{\sigma'(k)})$$

$$\times T_q(\omega_{k+1}, \ldots, \omega_{k+l})$$

$$= [k]!\, q^{R(\sigma_0(1),\ldots,\sigma_0(k+l))} sgn_q \sigma_0 Alt_q(S_q)(\omega_{\sigma'(1)} \cdots \omega_{\sigma'(k)})$$

$$\times T_q(\omega_{k+1}, \ldots, \omega_{k+l})$$

$$= 0$$

where we used the relation

$$
\begin{aligned}
& sgn_q(\sigma' \circ \sigma_0) \\
&= q^{-R(\sigma'\circ\sigma_0(1),\ldots,\sigma'\circ\sigma_0(k+l))} E^{1\cdots k+l}_{\sigma'\circ\sigma_0(1)\cdots\sigma'\circ\sigma_0(k+l)} \\
&= q^{R(\sigma_0(1),\ldots,\sigma_0(k+l))} q^{-R(\sigma'\circ\sigma_0(1),\ldots,\sigma'\circ\sigma_0(k+l))} E^{\sigma_0(1)\cdots\sigma_0(k+l)}_{\sigma'\circ\sigma_0(1)\cdots\sigma'\circ\sigma_0(k+l)} \\
&\quad \times q^{-R(\sigma_0(1),\ldots,\sigma_0(k+l))} E^{1\cdots k+l}_{\sigma_0(1)\cdots\sigma_0(k+l)} \\
&= q^{R(\sigma_0(1),\ldots,\sigma_0(k+l))} sgn_q\, \sigma_0\, sgn_q\, \sigma'
\end{aligned}
$$

Using Theorem 4, we can obtain the following theorem.

Theorem 5:

$$
\begin{aligned}
Alt_q(Alt_q(\omega \otimes \eta) \otimes \theta) &= Alt_q(\omega \otimes \eta \otimes \theta) \\
&= Alt_q(\omega \otimes Alt_q(\eta \otimes \theta))
\end{aligned}
$$

Proof:

$$
\begin{aligned}
& Alt_q(Alt_q(\eta \otimes \theta) - \eta \otimes \theta) \\
&\quad = Alt_q(\eta \otimes \theta) - Alt_q(\eta \otimes \theta) = 0
\end{aligned}
$$

Using Theorem 4, we have

$$
\begin{aligned}
0 &= Alt_q(\omega \otimes (Alt_q(\eta \otimes \theta) - \eta \otimes \theta)) \\
&= Alt_q(\omega \otimes Alt_q(\eta \otimes \theta)) - Alt_q(\omega \otimes \eta \otimes \theta)
\end{aligned}
$$

which completes the proof.

At last we reach the associativity of the q-wedge product.

Theorem 6. If $\omega \in \Lambda^k_q(V)$, $\eta \in \Lambda^l_q(V)$, and $\theta \in \Lambda^m_q(V)$, then

$$
\begin{aligned}
(\omega \wedge_q \eta) \wedge_q \theta &= \omega \wedge_q (\eta \wedge_q \theta) \\
&= \frac{[k+l+m]!}{[k]![l]![m]!} Alt_q(\omega \otimes \eta \otimes \theta)
\end{aligned}
$$

Proof:

$$
\begin{aligned}
& (\omega \wedge_q \eta) \wedge_q \theta \\
&= \frac{[k+l+m]!}{[k+l]![m]!} Alt_q((\omega \wedge_q \eta) \otimes \theta) \\
&= \frac{[k+l+m]!}{[k+l]![m]!} \frac{[k+l]!}{[k]![l]!} Alt_q(Alt_q(\omega \otimes \eta) \otimes \theta) \\
&= \frac{[k+l+m]!}{[k]![l]![m]!} Alt_q(\omega \otimes \eta \otimes \theta)
\end{aligned}
$$

which means the associativity of the q-deformed wedge product.

3. PROPERTY OF q-DEFORMED ALTERNATING TENSORS

In (5) the q-deformed wedge product for the q-deformed k-tensor ϕ and l-tensor ψ is defined, which is given by

$$\phi \wedge_q \psi = \frac{[k+l]!}{[k]![l]!} Alt_q(\phi \otimes \psi)$$
$$= \frac{1}{[k]![l]!} \sum_{\sigma \in S_{k+l}} sgn_q \sigma \phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)}) \times \psi(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)})$$

For $\phi, \psi \in \Lambda_q^1(V)$ we have

$$(\phi \wedge_q \psi)(v_1, v_2) = \phi(v_1)\psi(v_2) - q^{-2}\phi(v_2)\psi(v_1)$$

and

$$(\psi \wedge_q \phi)(v_1, v_2) = \psi(v_1)\phi(v_2) - q^{-2}\psi(v_2)\phi(v_1)$$

Consider another example. For $\phi \in \Lambda_q^1(V)$ and $\psi \in \Lambda_q^2(V)$, we have

$$\begin{aligned}
&(\phi \wedge_q \psi)(v_1, v_2, v_3) \\
&= \frac{1}{[2]} \{q^{-R(123)}E^{123}_{123}\phi(v_1)\psi(v_2, v_3) + q^{-R(132)}E^{123}_{132}\phi(v_1)\psi(v_3, v_2) \\
&\quad + q^{-R(213)}E^{123}_{213}\phi(v_2)\psi(v_1, v_3) \\
&\quad + q^{-R(231)}E^{123}_{231}\phi(v_2)\psi(v_3, v_1) + q^{-R(312)}E^{123}_{312}\phi(v_3)\psi(v_1, v_2) \\
&\quad + q^{-R(321)}E^{123}_{321}\phi(v_3)\psi(v_2, v_1)\} \\
&= \frac{1}{[2]} \{\phi(v_1)\psi(v_2, v_3) - q^{-2}\phi(v_1)\psi(v_3, v_2) - q^{-2}\phi(v_2)\psi(v_1, v_3) \\
&\quad + q^{-4}\phi(v_2)\psi(v_3, v_1) + q^{-4}\phi(v_3)\psi(v_1, v_2) - q^{-6}\phi(v_3)\psi(v_2, v_1)\} \\
&= \phi(v_1)\psi(v_2, v_3) - q^{-2}\phi(v_2)\psi(v_1, v_3) + q^{-4}\phi(v_3)\psi(v_1, v_2)
\end{aligned}$$

and

$$\begin{aligned}
&(\psi \wedge_q \phi)(v_1, v_2, v_3) \\
&= \frac{1}{[2]} \{\psi(v_1, v_2)\phi(v_3) - q^{-2}\psi(v_1, v_3)\phi(v_2) - q^{-2}\psi(v_2, v_1)\phi(v_3) \\
&\quad + q^{-4}\psi(v_2, v_3)\phi(v_1) + q^{-4}\psi(v_3, v_1)\phi(v_2) - q^{-6}\psi(v_3, v_2)\phi(v_1)\} \\
&= \psi(v_1, v_2)\phi(v_3) - q^{-2}\psi(v_1, v_3)\phi(v_2) + q^{-4}\psi(v_2, v_3)\phi(v_1)
\end{aligned}$$

where we used the q-alternating property of ψ. Here we have another expression for the q-deformed wedge product of $\phi \in \Lambda_q^k$ and $\psi \in \Lambda_q^l$:

$$(\phi \wedge_q \psi)(v_1, \ldots, v_{k+l}) = \sum_{\sigma^* \in S_{k+l}, \sigma^*(1)<\cdots<\sigma^*(k), \sigma^*(k+1)<\cdots<\sigma^*(k+l)} sgn_q\, \sigma^*\phi(v_{\sigma^*(1)}, \ldots, v_{\sigma^*(k)}) \times \psi(v_{\sigma^*(k+1)}, \ldots, v_{\sigma^*(k+l)}) \tag{12}$$

This is proved as follows:

$$\begin{aligned}
&(\phi \wedge_q \psi)(v_1, \ldots, v_{k+l}) \\
&= \frac{1}{[k]![l]!} \sum_{\sigma \in S_{k+l}} sgn_q \sigma\phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\psi(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \\
&= \frac{1}{[k]![l]!} \sum_{\sigma \in S_{k+l}} q^{-R(\sigma(1),\ldots,\sigma(k+l))} E^{1,\ldots,k+l}_{\sigma(1),\ldots,\sigma(k+l)} \\
&\quad \times \phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\psi(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \\
&= \frac{1}{[k]![l]!} \sum_{\sigma \in S_{k+l}} q^{-R(\sigma(1),\ldots,\sigma(k)) - R(\sigma(k+1),\ldots,\sigma(k+l)) - \Sigma_{i\in\{1,\ldots,k\}, j\in\{k+1,\ldots,k+l\}} R(\sigma(i),\sigma(j))} \\
&\quad \times E^{1,\ldots,k+l}_{\sigma(1),\ldots,\sigma(k+l)} \phi(v_{\sigma(1)}, \ldots, v_{\sigma(k)})\psi(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+l)}) \\
&= \frac{1}{[k]![l]!} \sum_{\sigma_1 \in S_k} q^{-R(\sigma_1(1),\ldots,\sigma_1(k))} \sum_{\sigma_2 \in S_l} q^{-R(\sigma_2(k+1),\ldots,\sigma_2(k+l))} \\
&\quad \times \sum_{\sigma^* \in S_{k+l}, \sigma^*(1)<\cdots<\sigma^*(k), \sigma^*(k+1)<\cdots<\sigma^*(k+l)} sgn_q \sigma^*\phi(v_{\sigma^*(1)}, \ldots, v_{\sigma^*(k)}) \\
&\quad \times \psi(v_{\sigma^*(k+1)}, \ldots, v_{\sigma^*(k+l)}) \\
&= \sum_{\sigma^* \in S_{k+l}, \sigma^*(1)<\cdots<\sigma^*(k), \sigma^*(k+1)<\cdots<\sigma^*(k+l)} sgn_q \sigma^*\phi(v_{\sigma^*(1)}, \ldots, v_{\sigma^*(k)}) \\
&\quad \times \psi(v_{\sigma^*(k+1)}, \ldots, v_{\sigma^*(k+l)})
\end{aligned}$$

where we used the following formula:

$$\sum_{\sigma \in S_k} q^{-R(\sigma(1),\ldots,\sigma(k))} = [k]!$$

Now we have the following property for $\phi \in \Lambda_q^k$ and $\psi \in \Lambda_q^l$:

$$\begin{aligned}
&(\phi \wedge_q \psi)(v_1, \ldots, v_{k+l}) \\
&\quad = (-q^{-2})^{kl}\hat{Q}(\psi \wedge_q \phi)(v_1, \ldots, v_{k+l})
\end{aligned}$$

where

$$\hat{Q}(\psi \wedge_q \phi)(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in S_{k+l}, \sigma(1) < \cdots < \sigma(l), \sigma(l+1) < \cdots < \sigma(k+l)} sgn_{q^{-1}} \sigma \psi(v_{\sigma(1)}, \ldots, v_{\sigma(l)}) \times \phi(v_{\sigma(l+1)}, \ldots, v_{\sigma(k+l)})$$

It is worth stressing that

$$\hat{Q}(\psi \wedge_q \phi)(v_1, \ldots, v_{k+l}) \neq \sum_{\sigma \in S_{k+l}} sgn_{q^{-1}} \sigma \psi(v_{\sigma(1)}, \ldots, v_{\sigma(l)}) \phi(v_{\sigma(l+1)}, \ldots, v_{\sigma(k+l)})$$

The proof is easy. Then we have

$$(\phi \wedge_q \psi)(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in S_{k+l}, \sigma(1) < \cdots < \sigma(k), \sigma(k+1) < \cdots < \sigma(k+l)} sgn_q \sigma \phi(v_{\sigma(1)}, \ldots, v_{(k+l)}) \times \psi(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+1)})$$

If we substitute

$$\sigma(i) = \sigma'(l+i) \quad \text{for} \quad i = 1, \ldots, k$$
$$\sigma(i) = \sigma'(i-k) \quad \text{for} \quad i = k+1, \ldots, k+l$$

we have

$$(\phi \wedge_q \psi)(v_1, \ldots, v_{k+l}) = \sum_{\sigma \in S_{k+l}, \sigma(1) < \ldots < \sigma(k), \sigma(k+1) < \ldots < \sigma(k+l)} sgn_q \, \sigma \phi(v_{\sigma'(l+1)}, \ldots, v_{\sigma'(k+l)} \psi(v_{\sigma'(1)}, \ldots, v_{\sigma'(l)})$$

If we write

$$sgn_q \sigma = q^{-R(\sigma(1), \ldots, \sigma(k+l))} s\hat{g}n_q \sigma$$

then we get

$$\begin{aligned} s\hat{g}n_q \sigma &= E^{1,\ldots,k+l}_{\sigma(1),\ldots,\sigma(k+l)} \\ &= E^{1,\ldots,k+l}_{\sigma(1),\ldots,\sigma(k+l)} E^{1,\ldots,k+l}_{\sigma'(1),\ldots,\sigma'(k+l)} E^{\sigma'(1),\ldots,\sigma'(k+l)}_{1,\ldots,k+l} \\ &= \frac{s\hat{g}n_{q^{-1}} \sigma'}{E_{\sigma(1),\ldots,\sigma(k),\sigma(k+1),\ldots,\sigma(k+l)} E_{\sigma'(1),\ldots,\sigma'(k),\sigma'(k+1),\ldots,\sigma'(k+l)}} \\ &= (-q^{-1})^{kl} s\hat{g}n_{q^{-1}} \sigma' \end{aligned}$$

where we used

$$E_{\sigma(1),\ldots,\sigma(k),\sigma(k+1),\ldots,\sigma(k+l)} = (-q)^{kl}$$

Its proof is given in Appendix B. On the other hand, we obtain

$$\begin{aligned}
&q^{-R(\sigma(1),\ldots,\sigma(k+l))} \\
&= q^{-R(\sigma'(1),\ldots,\sigma'(k))-R(\sigma'(k+1),\ldots,\sigma'(k+l))-\Sigma_{i\in\{1,\ldots,k\},j\in\{k+1,\ldots,k+l\}}R(\sigma'(i),\sigma'(j))} \\
&= q^{-\Sigma_{i\in\{1,\ldots,k\},j\in\{k+1,\ldots,k+l\}}R(\sigma'(i),\sigma'(j))} \\
&= q^{-\Sigma_{i\in\{1,\ldots,k\},j\in\{k+1,\ldots,k+l\}}(1-R(\sigma'(j),\sigma'(i)))} \\
&= q^{-kl}q^{\Sigma_{i\in\{1,\ldots,k\},j\in\{k+1,\ldots,k+l\}}R(\sigma'(j),\sigma'(i))} \\
&= q^{-kl}q^{R(\sigma'(1),\ldots,\sigma'(l),\sigma'(l+1),\ldots,\sigma'(k+l))}
\end{aligned}$$

Thus we have

$$sgn_q\sigma = (-q^{-2})^{kl}sgn_{q^{-1}}\sigma'$$

Therefore we obtain

$$\begin{aligned}
&(\phi \wedge_q \psi)(v_1, \ldots, v_{k+l}) \\
&= (-q^{-2})^{kl} \sum_{\sigma'\in S_{k+l},\sigma'(1)<\cdots<\sigma'(l),\sigma'(l+1)<\cdots<\sigma'(k+l)} sgn_{q^{-1}}\sigma'\psi(v_{\sigma'(1)}, \ldots, v_{\sigma'(l)}) \\
&\quad \times \phi(v_{\sigma'(l+1)}, \ldots, v_{\sigma'(k+l)})
\end{aligned}$$

which completes the proof.

4. q-DEFORMED STOKES THEOREM

In this section we discuss the q-deformed Stokes theorem and prove it. Let a q-deformed singular n-cube in $A \subset R^n$ be a function c: $[0, 1]^n \to A$.

A particular example of a q-deformed singular n-cube in R^n is a q-deformed standard n-cube I^n: $[0, 1]^n \to R^n$ defined by $I^n(x) = x$ for $x \in [0, 1]^n$. Let the q-deformed n-chain in A be the finite formal sum of q-deformed singular n-cubes in A multiplied by integers. For each q-deformed singular n-chain c in A we define a q-deformed $(n - 1)$-chain in A called the q-boundary of c and denoted ∂c. For each i with $1 \le i \le n$ we define two q-deformed $(n - 1)$-cubes $I^n_{(i,0)}$ and $I^n_{(i,1)}$ as follows.

If $x \in [0, 1]^{n-1}$, then we have

$$\begin{aligned}
I^n_{(i,0)}(x) &= I^n(x^1, \ldots, x^{i-1}, 0, x^i, \ldots, x^{n-1}) \\
&= (x^1, \ldots, x^{i-1}, 0, x^i, \ldots, x^{n-1})
\end{aligned}$$

$$I^n_{(i,1)}(x) = I^n(x^1, \ldots, x^{i-1}, 1, x^i, \ldots, x^{n-1})$$
$$= (x^1, \ldots, x^{i-1}, 1, x^i, \ldots, x^{n-1})$$

We will call $I^n_{(i,\alpha)}$ the q-deformed (i, α)-face of I^n. We then define the q-boundary of I^n as

$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} q^{i-1} I^n_{(i,\alpha)}$$

For a general q-deformed singular n-cube c: $[0, 1]^n \to A$ we first define the q-deformed (i, α)-face

$$c_{(i,\alpha)} = q^{-i} c \circ I^n_{(i,\alpha)}$$

and then define the q-boundary of c as

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} q^{i-1} c_{(i,\alpha)}$$

Similarly we can define the q-deformed (j, β)-face of the q-deformed standard $(n - 1)$-cube $I^n_{(i,\alpha)}$ by

$$(c_{(i,\alpha)})_{(j,\beta)} = q^{-i-j} c \circ (I^n_{(i,\alpha)})_{(j,\beta)}$$

Theorem 7. If c is a q-deformed n-chain in A, then $\partial(\partial c) = 0$. Briefly, $\partial^2 = 0$.

Proof. Let $i \le j$ and consider $(I^n_{(i,\alpha)})_{(j,\beta)}$. If $x \in [0, 1]^{n-2}$, then we obtain

$$\begin{aligned}
&(I^n_{(i,\alpha)})_{(j,\beta)}(x)\\
&= I^n_{(i,\alpha)}(I^{n-1}_{(j,\beta)})(x)\\
&= I^n_{(i,\alpha)}(x^1, \ldots, x^{j-1}, \beta, x^j, \ldots, x^{n-2})\\
&= (x^1, \ldots, x^{i-1}, \alpha, x^i, \ldots, x^{j-1}, \beta, x^j, \ldots, x^{n-2})
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&(I^n_{(j+1,\beta)})_{(i,\alpha)}(x)\\
&= I^n_{(j+1,\beta)}(I^{n-1}_{(i,\alpha)})(x)\\
&= I^n_{(j+1,\beta)}(x^1, \ldots, x^{i-1}, \alpha, x^i, \ldots, x^{n-2})\\
&= (x^1, \ldots, x^{i-1}, \alpha, x^i, \ldots, x^{j-1}, \beta, x^j, \ldots, x^{n-2})
\end{aligned}$$

Thus we have

$$(I^n_{(i,\alpha)})_{(j,\beta)} = (I^n_{(j+1,\beta)})_{(i,\alpha)} \qquad \text{for} \quad i \leq j$$

It follows easily for any q-deformed singular n-cube c that

$$(c_{(i,\alpha)})_{(j,\beta)} = q(c_{(j+1,\beta)})_{(i,\alpha)} \qquad \text{for} \quad i \leq j$$

Then we get

$$\begin{aligned}
\partial^2 c &= \partial\left(\sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} q^{i-1} c_{(i,\alpha)}\right) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n-1} \sum_{\alpha,\beta=0,1} (-1)^{i+j+\alpha+\beta} q^{i+j-2} (c_{(i,\alpha)})_{(j,\beta)} \\
&= \sum_{i\leq j} \sum_{\alpha,\beta=0,1} (-1)^{i+j+\alpha+\beta} q^{i+j-2} (c_{(i,\alpha)})_{(j,\beta)} \\
&\quad + \sum_{i>j} \sum_{\alpha,\beta=0,1} (-1)^{i+j+\alpha+\beta} q^{i+j-2} (c_{(i,\alpha)})_{(j,\beta)}
\end{aligned}$$

The first term is rewritten in the form

$$\text{first term} = \sum_{i=1}^{n-1} \sum_{j=1, i\leq j}^{n-1} \sum_{\alpha,\beta=0,1} (-1)^{i+j+\alpha+\beta} q^{i+j-1} (c_{(j+1,\beta)})_{(i,\alpha)}$$

The second term is written as

$$\begin{aligned}
&\text{second term} \\
&= \sum_{i=2}^{n} \sum_{j=1, i>j}^{n-1} \sum_{\alpha,\beta=0,1} (-1)^{i+j+\alpha+\beta} q^{i+j-2} (c_{(i,\alpha)})_{(j,\beta)} \\
&= \sum_{i=1}^{n-1} \sum_{j=2, i<j}^{n} \sum_{\alpha,\beta=0,1} (-1)^{i+j+\alpha+\beta} q^{i+j-2} (c_{(j,\alpha)})_{(i,\beta)} \\
&= -\sum_{i=1}^{n-1} \sum_{j=1, i\leq j}^{n-1} \sum_{\alpha,\beta=0,1} (-1)^{i+j+\alpha+\beta} q^{i+j-1} (c_{(j+1,\beta)})_{(i,\alpha)}
\end{aligned}$$

Comparing the first term with second term, we find that

$$\partial^2 c = 0$$

which indicates that

$$\partial^2 = 0$$

Theorem 8:

$$\int_c d\omega = \int_{\partial c} \omega$$

Proof. Suppose that $C = I^k$ and ω is a q-deformed $(k-1)$-form on $[0, 1]^k$. Then ω is the sum of q-deformed $(k-1)$-forms of the type

$$f\,dx^1 \wedge_q \cdots \wedge_q \hat{dx^i} \wedge_q \cdots \wedge_q dx^k$$

Note that

$$\int_{[0,1]^{k-1}} I^k_{(j,\alpha)}{}^*(f\,dx^1 \wedge_q \cdots \wedge_q \hat{dx^i} \wedge_q \cdots \wedge_q dx^k)$$

$$= \delta_{ij} \int_{[0,1]^k} f(x^1, \ldots, \alpha, \ldots, x^k)\,dx^1 \cdots dx^k$$

Therefore we have

$$\int_{[0,1]^k} f\,dx^1 \wedge_q \cdots \wedge_q \hat{dx^i} \wedge_q \cdots \wedge_q dx^k$$

$$= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} q^{j-1} \int_{I^k_{(j,\alpha)}} f\,dx^1 \wedge_q \cdots \wedge_q \hat{dx^i} \wedge_q \cdots \wedge_q dx^k$$

$$= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} q^{j-1} \int_{[0,1]^{k-1}} I^k_{(j,\alpha)}{}^*(f\,dx^1 \wedge_q \cdots \wedge_q \hat{dx^i} \wedge_q \cdots \wedge_q dx^k)$$

$$= \sum_{j=1}^{k} \sum_{\alpha=0,1} (-1)^{j+\alpha} q^{j-1} \delta_{ij} \int_{[0,1]^k} f(x^1, \ldots, \alpha, \ldots, x^k)\,dx^1 \cdots dx^k$$

$$= (-1)^{i+1} q^{i-1} \int_{[0,1]^k} f(x^1, \ldots, 1, \ldots, x^k)\,dx^1 \cdots dx^k$$

$$+ (-1)^{i} q^{i-1} \int_{[0,1]^k} f(x^1, \ldots, 0, \ldots, x^k)\,dx^1 \cdots dx^k$$

$$= (-q)^{i-1} \int_{[0,1]^k} [f(x^1, \ldots, 1, \ldots, x^k) - f(x^1, \ldots, 0, \ldots, x^k)]\,dx^1 \cdots dx^k$$

On the other hand, we have

$$\int_{I^k} d(f\,dx^1 \wedge_q \cdots \wedge_q \hat{dx^i} \wedge_q \cdots \wedge_q dx^k)$$

$$= \int_{I^k} D_i f\,dx^i \wedge_q dx^1 \wedge_q \cdots \wedge_q \hat{dx^i} \wedge_q \cdots \wedge_q dx^k$$

$$= \int_{I^k} (-q)^{i-1} D_i f\,dx^1 \wedge_q \cdots \wedge_q dx^k$$

$$= \int_{[0,1]^k} (-q)^{i-1} D_i f \, dx^1 \cdots dx^k$$

$$= (-q)^{i-1} \int_{[0,1]^{k-1}} [f(x^1, \ldots, 1, \ldots, x^k) - f(x^1, \ldots, 0, \ldots, x^k)]$$
$$\times dx^1 \cdots \hat{d}x^i \cdots dx^k$$

$$= (-q)^{i-1} \int_{[0,1]^{k-1}} [f(x^1, \ldots, 1, \ldots, x^k) - f(x^1, \ldots, 0, \ldots, x^k)]$$
$$\times dx^1 \cdots dx^k$$

where D_i means $\partial/\partial x^i$. We have

$$\int_{I^k} d\omega = \int_{\partial I^k} \omega$$

From the above result we can easily extend the result for I^k to the arbitrary q-deformed singular k-chain c. Thus we have

$$\int_c d\omega = \int_{\partial c} \omega$$

5. q-DEFORMED POINCARÉ LEMMA

In this section we will prove the following lemma, called the q-deformed Poincaré lemma.

Theorem (Poincaré lemma). Every q-closed, q-deformed form is q-exact.[3]

Proof. Let the q-deformed 1-form ω be given by

$$\omega = \sum_{i_1 < \cdots < i_l} \omega_{i_1, \ldots, i_l} dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_l}$$

We will show that

$$\omega = I(d\omega) + d(I\omega)$$

[3] A q-deformed tensor ϕ is said to be q-closed if $d\phi = 0$; a q-deformed tensor ϕ is said to be q-exact if there exists some q-deformed tensor η such that $\phi = d\eta$.

which indicates that $d\omega = 0$ leads to $\omega = d(I\omega)$. We define the operator I on $\omega(x)$ as

$$I\omega(x) = \sum_{i_1<\cdots<i_l} \sum_{\alpha=1}^{l} (-q)^{-(\alpha-1)}\left(\int_0^1 t^{l-1}\omega_{i_1\cdots i_l}(tx)\,dt\right) \times x^{i_\alpha}dx^{i_1} \wedge_q \cdots \wedge_q \hat{dx}^{i_\alpha} \wedge_q \cdots \wedge_q dx^{i_l}$$

Acting with the exterior derivative d on $I\omega$, we obtain

$$\begin{aligned} d(I\omega) &= \sum_{i_1<\cdots<i_l} \sum_{\alpha=1}^{l} (-q)^{-(\alpha-1)}\left(\int_0^1 t^{l-1}\omega_{i_1\cdots i_l}(tx)\,dt\right) \\ &\quad \times dx^{i_\alpha} \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q \hat{dx}^{i_\alpha} \wedge_q \cdots \wedge_q dx^{i_l} \\ &\quad + \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-q)^{-(\alpha-1)}\left(\int_0^1 t^{l}(D_j\omega_{i_1\cdots i_l})(tx)\,dt\right) \\ &\quad \times dx^j x^{i_\alpha} \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q \hat{dx}^{i_\alpha} \wedge_q \cdots \wedge_q dx^{i_l} \\ &= \sum_{i_1<\cdots<i_l} \sum_{\alpha=1}^{l} \left(\int_0^1 t^{l-1}\omega_{i_1\cdots i_l}(tx)\,dt\right) dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_l} \\ &\quad + \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-q)^{-(\alpha-1)}\left(\int_0^1 t^{l}(D_j\omega_{i_1\cdots i_l})(tx)\,dt\right) \\ &\quad \times dx^j x^{i_\alpha} \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q \hat{dx}^{i_\alpha} \wedge_q \cdots \wedge_q dx^{i_l} \\ &= l \sum_{i_1<\cdots<i_l} \left(\int_0^1 t^{l-1}\omega_{i_1\cdots i_l}(tx)\,dt\right) dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_l} \\ &\quad + \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-q)^{-(\alpha-1)}\left(\int_0^1 t^{l}(D_j\omega_{i_1\cdots i_l})(tx)\,dt\right) \\ &\quad \times dx^j\, x^{i_\alpha} \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q \hat{dx}^{i_\alpha} \wedge_q \cdots \wedge_q dx^{i_l} \end{aligned}$$

where the caret over dx^{i_α} indicates that it is deleted, and D_j means derivative with respect to x^j. On the other hand, we have

$$\begin{aligned} d\omega &= \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} D_j\omega_{i_1\cdots i_l} dx^j \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q \cdots \wedge_q dx^{i_l} \\ &= \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-q)^{-(\alpha-1)} D_j\omega_{i_1\cdots i_l} \\ &\quad \times dx^j \wedge_q dx^{i_\alpha} \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q \hat{dx}^{i_\alpha} \wedge_q \cdots \wedge_q dx^{i_l} \end{aligned}$$

$$= \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-q)^{-(\alpha-1)} D_j \omega_{i_1\cdots i_l} E^{ji_\alpha}_{i_\alpha j}$$
$$\times\, dx^{i_\alpha} \wedge_q dx^j \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q \hat{dx}^{i_\alpha} \wedge_q \cdots \wedge_q dx^{i_l}$$

where we used the formula

$$dx^i \wedge_q dx^j = E^{ij}_{ji}\, dx^j \wedge_q dx^i$$

and

$$E^{ij}_{ji} = \frac{E_{ij}}{E_{ji}}$$

Acting with the operator I on $d\omega$ leads to

$$I(d\omega) = \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} \left(\int_0^1 t^l (D_j \omega_{i_1\cdots i_l})(tx)\, dt \right) x^j\, dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_l}$$
$$+ \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-q)^{-(\alpha-1)} E^{ji_\alpha}_{i_\alpha j} \left(\int_0^1 t^l (D_j \omega_{i_1\cdots i_l})(tx)\, dt \right)$$
$$\times\, x^{i_\alpha}\, dx^j \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q \hat{dx}^{i_\alpha} \wedge_q \cdots \wedge_q dx^{i_l}$$

Thus we have

$$d(I\omega) + I(d\omega)$$
$$= l \sum_{i_1<\cdots<i_l} \left(\int_0^1 t^{l-1} \omega_{i_1\cdots i_l}(tx)\, dt \right) dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_l}$$
$$+ \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} \left(\int_0^1 t^l (D_j \omega_{i_1\cdots i_l})(tx)\, dt \right) x^j\, dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_l}$$
$$+ \sum_{i_1<\cdots<i_l} \sum_{j=1}^{n} \sum_{\alpha=1}^{l} (-q)^{-(\alpha-1)} \left(\int_0^1 t^l (D_j \omega_{i_1\cdots i_l})(tx)\, dt \right)$$
$$\times\, (dx^j x^{i_\alpha} + E^{ji_\alpha}_{i_\alpha j} x^{i_\alpha} dx^j) \wedge_q dx^{i_1} \wedge_q \cdots \wedge_q \hat{dx}^{i_\alpha} \wedge_q \cdots \wedge_q dx^{i_l}$$

Using the relation

$$dx^j x^{i_\alpha} = -E^{ji_\alpha}_{i_\alpha j} x^{i_\alpha} dx^j$$

we have

$$
\begin{aligned}
d(I\omega) &+ I(d\omega) \\
&= \sum_{i_1<\cdots<i_l} \int_0^1 \left(l t^{l-1} \omega_{i_1\cdots i_l}(tx) + \sum_{j=1}^{n} t^l D_j \omega_{i_1\cdots i_l}(tx) x^j \right) dt \\
&\quad \times dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_l} \\
&= \sum_{i_1<\cdots<i_l} \int_0^1 \frac{d}{dt} (t^l \omega_{i_1\cdots i_l}(tx))\, dt\, dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_l} \\
&= \sum_{i_1<\cdots<i_l} \omega_{i_1,\ldots,i_l}(x)\, dx^{i_1} \wedge_q \cdots \wedge_q dx^{i_l} \\
&= \omega(x)
\end{aligned}
$$

Therefore we reach the relation

$$\omega = d(I\omega) + I(d\omega)$$

which completes the proof.

6. CONCLUSIONS

In this paper we have discussed more or less mathematical topics in q-deformed physics. In $q \to 1$ this theory goes back to the ordinary differential form theory. The alternating tensor has the same form as the antisymmetric states in many-body quantum mechanics. In this sense we can guess that we will use the theory of q-deformed alternating tensors to construct the q-symmetric state related to the q-boson algebra. We hope that the theorems given in this paper will be widely used in developing q-deformed physics and mathematics.

APPENDIX A

In this paper we prove the useful identity

$$\sum_{\sigma \in S_k} q^{-R(\sigma(1)\cdots\sigma(k))} = [k]!$$

In order to prove the above identity, we use mathematical induction; then we assume that the identity holds for k. For $k + 1$ we have

$$
\begin{aligned}
&\sum_{\sigma \in S_{k+1}} q^{-R(\sigma(1),\ldots,\sigma(k),\sigma(k+1))} \\
&\quad = \sum_{\sigma \in S_k, \sigma(k+1)=k+1} q^{-R(\sigma(1),\ldots,\sigma(k),\sigma(k+1)=k+1))} \\
&\qquad + \sum_{i=1}^{k} \sum_{\sigma \in S_k, \sigma(i)=k+1} q^{-R(\sigma(1),\ldots,\sigma(i-1),\sigma(i)=k+1,\sigma(i+1),\ldots,\sigma(k+1))}
\end{aligned}
$$

In the first term of the right-hand side of the above equation, we have, from the definition of R,

$$R(\sigma(1), \ldots, \sigma(k), \sigma(k+1) = k+1) = R(\sigma(1), \ldots, \sigma(k))$$

Hence the first term equals to $[k]!$ by assumption. In the second term we get

$$\begin{aligned} &R(\sigma(1), \ldots, \sigma(i-1), \sigma(i) = k+1, \sigma(i+1), \ldots, \sigma(k+1)) \\ &\quad = R(\sigma(1), \ldots, \sigma(i-1), \sigma(i) = k+1) + R(\sigma(i+1), \ldots, \sigma(k+1)) \\ &\qquad + \sum_{a=1,\ldots,i,b=i+1,\ldots,k+1} R(\sigma(a), \sigma(b)) \\ &\quad = R(\sigma(1), \ldots, \sigma(i-1)) + R(\sigma(i+1), \ldots, \sigma(k+1)) \\ &\qquad + \sum_{a=1,\ldots,i-1,b=i+1,\ldots,k+1} R(\sigma(a), \sigma(b)) + \sum_{b=i+1,\ldots,k+1} R(k+1, \sigma(b)) \end{aligned}$$

Since $k+1$ is always larger than the $\sigma(b)$'s, the last term gives $k-i+1$. Then we have

$$\begin{aligned} &R(\sigma(1), \ldots, \sigma(i-1), \sigma(i) = k+1, \sigma(i+1), \ldots, \sigma(k+1)) \\ &\quad = R(\sigma(1), \ldots, \sigma(i-1), \hat{\sigma}(i), \sigma(i+1), \ldots, \sigma(k+1)) + k - i + 1 \end{aligned}$$

where $\hat{\sigma}(i)$ means that $\sigma(i)$ is deleted. Therefore we have

$$\begin{aligned} &\sum_{\sigma \in S_{k+1}} q^{-R(\sigma(1),\ldots,\sigma(k+1))} \\ &\quad = \left(1 + \sum_{i=1}^{k} q^{-(k-i+1)}\right)[k]! = [k+1]! \end{aligned}$$

which implies that the identity holds for $k+1$. By mathematical induction, we can say that identity holds for all natural numbers k.

APPENDIX B

In this appendix we prove the formula

$$E_{i_1 \cdots i_l j_1 \cdots j_{i_k}} E_{j_1 \cdots j_k i_1 \cdots i_l} = (-q)^{kl}$$

where

$$i_1 < i_2 < \cdots < i_l$$
$$j_1 < j_2 < \cdots < j_k$$

We assume that the above relation holds. Then we only have to prove that

$$E_{i_1 \cdots i_l i_{l+1} j_1 \cdots j_k} E_{j_1 \cdots j_k i_1 \cdots i_l i_{l+1}} = (-q)^{k(l+1)}$$

and

$$E_{i_1\cdots i_l j_1\cdots j_k j_{k+1}} E_{j_1\cdots j_k j_{k+1} i_1\cdots i_l} = (-q)^{(k+1)l}$$

First we will prove the first identity. For $i_{l+1} = l + k + 1$ we have

$$\begin{aligned}\text{LHS of first identity} &= (-q)^k(-q)^{kl}\\ &= (-q)^{k(l+1)}\end{aligned}$$

For $i_{l+1} \neq l + k + 1$ we get

$$i_1 < i_2 < \cdots < i_l < i_{l+1}$$

$$j_1 < j_2 < \cdots < j_k$$

so we have $j_k = l + k + 1$; then we get

$$\begin{aligned}\text{LHS of first identity} &= (-q)^{l+1} E_{i_1\cdots i_{l+1} j_1\cdots j_{k-1}} E_{j_1\cdots j_{k-1} i_1\cdots i_{l+1}}\\ &= (-q)^{l+1}(-q)^{(l+1)(k-1)}\\ &= (-q)^{(l+1)k}\end{aligned}$$

Now we will prove the second identity. For $j_k = k + l + 1$ we have

$$\text{LHS of second identity} = (-q)^l(-q)^{kl} = (-q)^{l(k+1)}$$

For $j_{k+1} \neq l + k + 1$, we have $i_l = l + k + 1$, so

$$\begin{aligned}\text{LHS of second identity} &= (-q)^{k+1} E_{i_1\cdots i_{l-1} j_1\cdots j_{k+1}} E_{j_1\cdots j_{k+1} i_1\cdots i_{l-1}}\\ &= (-q)^{k+1}(-q)^{(l-1)(k+1)}\\ &= (-q)^{l(k+1)}\end{aligned}$$

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